# Chapter 5

# Scaling and transformation groups. Renormalization group

## 5.1 Dimensional analysis and transformation groups

We recall the definition of a transformation group. Suppose we have a set of transformations with $k$ parameters,

$$x'_\nu = f_\nu(x_1, \dots, x_n; A_1, \dots, A_k), \qquad \nu = 1, \dots, n, \tag{5.1}$$

where the $f_\nu$ are smooth functions of their arguments in a certain domain. We say that this set forms a $k$-parameter group of transformations if the following conditions are satisfied.

1. Among the transformations (5.1) there exists the identity transformation.
2. For each transformation of the set (5.1) there exists an inverse transformation that also belongs to the set (5.1).
3. For each pair of transformations of the set (5.1), i.e. a transformation **A** with parameters $A_1, \dots, A_k$ and a transformation **B** with parameters $B_1, \dots, B_k$, a transformation **C** with parameters $C_1, \dots, C_k$, which also belongs to the set (5.1), exists and is uniquely determined such that successive realization of the transformations **A** and **B** is equivalent to the transformation **C**. The transformation **C** is called the *product* of the transformations **A** and **B**.

Dimensional analysis, which was considered in detail in Chapter 1, has a transparently group-theoretical nature. Group considerations can turn out to be useful also in those cases where dimensional analysis alone becomes insufficient to establish scaling laws and the self-similarity of a solution and to determine self-similar variables. A special place belongs here to the *renormalization group*, a concept now popular in theoretical physics.

Dimensional analysis is based on the $\Pi$-theorem (see Chapter 1). This theorem allows one to express a dimensional, generally speaking, function of



$n = k + m$ dimensional governing parameters, i.e. the physically meaningful relationship

$$a = f(a_1, \dots, a_k, b_1, \dots, b_m) \tag{5.2}$$

where $a_1, \dots, a_k$ are the governing parameters with independent dimensions, as a dimensionless function of $m$ dimensionless parameters

$$\Pi = \Phi(\Pi_1, \dots, \Pi_m),$$

where

$$\Pi = \frac{a}{a_1^p \cdots a_k^r}, \qquad \Pi_1 = \frac{b_1}{a_1^{p_1} \cdots a_k^{r_1}}, \qquad \dots, \qquad \Pi_m = \frac{b_m}{a_1^{p_m} \cdots a_k^{r_m}}.$$

This means that the function $f$ in (5.2) possesses the property of generalized homogeneity:

$$f(a_1, \dots, a_k, b_1, \dots, b_m) = a_1^p \cdots a_k^r \, \Phi\left(\frac{b_1}{a_1^{p_1} \cdots a_k^{r_1}}, \dots, \frac{b_m}{a_1^{p_m} \cdots a_k^{r_m}}\right).$$

We note now that, for any positive numbers $A_1, \dots, A_k$, the scaling transformation of the governing parameters with independent dimensions

$$a'_1 = A_1 a_1, \qquad a'_2 = A_2 a_2, \qquad \dots, \qquad a'_k = A_k a_k \tag{5.3}$$

can be obtained by changing from the original system of units to some other system belonging to the same class. At the same time the values of the remaining parameters $a, b_1, \dots, b_m$ vary in accordance with their dimensions:

$$\begin{aligned} a' &= A_1^p \cdots A_k^r a, \\ b'_1 &= A_1^{p_1} \cdots A_k^{r_1} b_1, \\ &\vdots \\ b'_m &= A_1^{p_m} \cdots A_k^{r_m} b_m. \end{aligned} \tag{5.4}$$

Direct verification shows easily that the transformations (5.3), (5.4) form a $k$-parameter group. Indeed, if $A_1 = A_2 = \cdots = A_k = 1$ then the transformation (5.3), (5.4) becomes an identity transformation. For each transformation **A** in the set (5.3), (5.4) there exists an inverse transformation **B** with parameter values

$$B_1 = \frac{1}{A_1}, \qquad B_2 = \frac{1}{A_2}, \qquad \dots, \qquad B_k = \frac{1}{A_k}$$

which also belongs to the set (5.3), (5.4) and such that the successive realization of transformations **A** and **B** returns the variables to their original values. For each pair **A, B** of transformations (5.3), (5.4), with parameter values $A_1, \dots, A_k$ and $B_1, \dots, B_k$, there exists one and only one transformation **C**, with parameter values $C_1 = A_1 B_1, C_2 = A_2 B_2 \dots, C_k = A_k B_k$, also belonging to the class

(5.3), (5.4) and such that the successive realization of transformations **A** and **B** is equivalent to the transformation **C**.

The quantities $\Pi, \Pi_1, \ldots, \Pi_m$ remain unchanged for all transformations of the group (5.3), (5.4), i.e. they are *invariants* of this group. Thus, the Π-theorem is a simple consequence of the covariance principle: relations with a physical meaning among dimensional quantities of the form (5.2) can be represented in a form invariant with respect to the group of similarity transformations of the governing parameters with independent dimensions (5.3), (5.4), each transformation corresponding to a transition to a different system of units (within a given class). The number of independent invariants of the group is less than the total number of governing parameters by the number $k$ of parameters of the group.

The invariance of the formulation, and hence the solution, of any physically meaningful problem with respect to the group of transformations (5.3), (5.4) is thus necessary according to the general physical covariance principle. It can turn out, however, that there exists a richer group with respect to which the formulation of the special problem considered is invariant. Then the number of arguments of the function $\Phi$ in the universal (invariant) relation obtained after applying the Π-theorem in its own right should be reducible by the number of parameters of the supplementary group. Here the solution can turn out to be self-similar, and the self-similar variables can be determined as a result of using the invariance with respect to the supplementary group, although this self-similarity is not implied by dimensional analysis (which exploits invariance with respect to the group of similarity transformations of the governing parameters with independent dimensions). We consider below an instructive example that will clarify this idea.

## 5.2 Problem: the boundary layer on a flat plate in a uniform flow

The problem of steady viscous incompressible flow past a semi-infinite flat plate placed along a uniform stream (Figure 5.1) leads to a system of Navier–Stokes equations and the equation of continuity (see Batchelor 1967; Germain 1986; Landau and Lifshitz 1987):

$$
\begin{aligned}
u\partial_x u + v\partial_y u &= -\frac{1}{\rho}\,\partial_x p + \nu\left(\partial^2_{xx}u + \partial^2_{yy}u\right),\\
u\partial_x v + v\partial_y v &= -\frac{1}{\rho}\,\partial_y p + \nu\left(\partial^2_{xx}v + \partial^2_{yy}v\right),\\
\partial_x u + \partial_y v &= 0.
\end{aligned}
\qquad (5.5)
$$

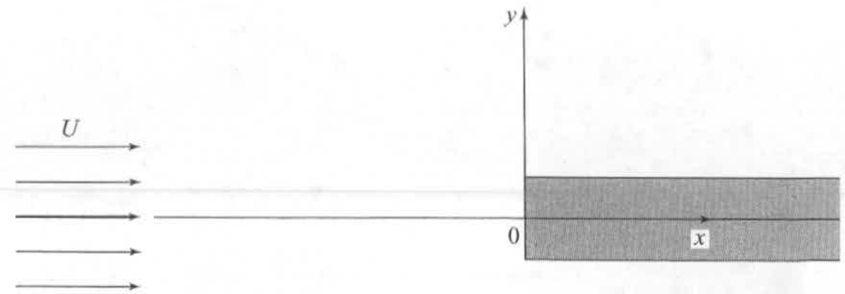


Figure 5.1. Viscous flow past a thin semi-infinite plate.

Here $x$ and $y$ are the longitudinal and transverse Cartesian coordinates, $u(x, y)$ and $v(x, y)$ are the corresponding velocity components, $p$ is the pressure, $\nu$ is the kinematic viscosity coefficient and $\rho$ is the density of the fluid.

The boundary conditions for the problem under consideration can be represented in the form

$$
\begin{aligned}
&u(x,0) = v(x,0) = 0, && x \geq 0,\\
&u(x,y) \to U, \quad v(x,y) \to 0 && \text{for } y^2 \to \infty \text{ and arbitrary } x\\
& && \text{and for } x \to -\infty \text{ and arbitrary } y.
\end{aligned}
$$

Here $U$ is the constant speed of the uniform exterior flow; the origin of coordinates $x = 0$, $y = 0$ corresponds to the tip of the plate. Up to now no single problem of viscous flow past a body has been solved analytically; the problem of the flow past a semi-infinite plate presented above, in spite of its seeming simplicity, does not constitute an exception.

At the beginning of the last century Prandtl (1904) proposed the idea of the *boundary layer*, which revolutionized fluid mechanics as a whole and, in particular, led to an asymptotic approximate analytic solution of the problem of viscous flow past a plate. This solution was obtained by Prandtl's student Blasius (1908) and modified by Toepfer (1912). The basic model of Prandtl in application to this problem is that at large Reynolds number the effects of viscosity are concentrated in a thin layer surrounding the plate only. Prandtl's hypothesis and certain qualitative considerations allowed a reduction of the model to a simplified one (see Batchelor 1967; Schlichting 1968; Germain 1986; Landau and Lifshitz 1987), the system of equations

$$
u\partial_x u + v\partial_y u = \nu\partial^2_{yy}u, \qquad \partial_x u + \partial_y v = 0 \qquad (5.6)
$$

under boundary conditions at $x > 0$, $y > 0$

$$
u(0,y) = U, \qquad u(x,\infty) = U, \qquad u(x,0) = v(x,0) = 0. \qquad (5.7)
$$

Two comments: first, no one has been able, up to now, to give a rigorous mathematical derivation of the system (5.6), (5.7) from the Navier–Stokes equations at large Reynolds numbers without additional assumptions – this system remains a result of Prandtl's intuition. The second comment concerns the second of the boundary conditions (5.7), which seems paradoxical: it is claimed that the boundary layer is thin yet the condition is taken at infinity. In fact, this paradox is explained by the asymptotic character of the qualitative derivation of the system (5.6), (5.7). This derivation is based on a 'stretching' of the system of coordinates, an asymptotic analysis of the problem in the stretched coordinates and a subsequent return to the original coordinates. This asymptotic procedure is illuminated by an original example proposed by Friedrichs (1966).

We apply to the problem in the boundary-layer approximation (5.6), (5.7) the standard procedure of dimensional analysis. The governing parameters are $\nu$, $x$, $U$ and $y$, so that

$$u = f_u(\nu, x, U, y), \qquad v = f_v(\nu, x, U, y). \tag{5.8}$$

The dimensions of the involved quantities are

$$[u] = [v] = [U] = \frac{L}{T}, \qquad [x] = [y] = L, \qquad [\nu] = \frac{L^2}{T} \tag{5.9}$$

so that, according to the standard procedure of dimensional analysis,

$$\Pi_u = \frac{u}{U} = \Phi_u(\Pi_1, \Pi_2), \qquad \Pi_v = \frac{v}{U} = \Phi_v(\Pi_1, \Pi_2). \tag{5.10}$$

Here

$$\Pi_1 = \xi = \frac{Ux}{\nu}, \qquad \Pi_2 = \eta = \frac{Uy}{\nu}. \tag{5.11}$$

In the new variables the relations (5.6), (5.7) are reduced to the form

$$\Phi_u \partial_\xi \Phi_u + \Phi_v \partial_\eta \Phi_u = \partial^2_{\eta\eta} \Phi_u, \qquad \partial_\xi \Phi_u + \partial_\eta \Phi_v = 0$$
$$\Phi_u(0, \eta) = \Phi_u(\xi, \infty) = 1, \qquad \Phi_u(\xi, 0) = \Phi_v(\xi, 0) = 0 \tag{5.12}$$

We see that the direct application of dimensional analysis does not give any simplification of the problem. In fact, the only distinction between the system (5.6), (5.7) and the system (5.12) is that in the latter the constants $\nu$ and $U$ are equal to unity: a purely cosmetic transformation.

It is instructive, however, that the system (5.12) is invariant with respect to an additional transformation group. Indeed, let $\Phi_u(\xi, \eta)$, $\Phi_v(\xi, \eta)$ be a solution of the system (5.12) which exists and is unique. Let us consider a one-parameter transformation group:

$$\xi' = \alpha^2 \xi, \qquad \eta' = \alpha\eta;$$
$$\Phi'_u(\xi', \eta') = \Phi_u(\xi, \eta), \qquad \Phi'_v(\xi', \eta') = \alpha^{-1}\Phi_v(\xi, \eta), \tag{5.13}$$

where $\alpha > 0$ is the parameter. It is easy to verify by direct substitution that the set of transformations (5.13) is a group: $\alpha = 1$ gives the identical transformation, $\beta = \alpha^{-1}$ gives the transformation inverse to $\alpha$ and $\gamma = \alpha\beta$ gives the product of the transformations with parameters $\alpha$ and $\beta$. Substituting (5.13) into (5.12), we obtain for arbitrary positive $\alpha$ the same problem as (5.12) but in the variables $\xi', \eta', \Phi'_u, \Phi'_v$. In view of the uniqueness requirement, the solution $\Phi'_u, \Phi'_v$ should also be unique, so that

$$\Phi_u(\xi, \eta) = \Phi'_u(\xi', \eta') = \Phi_u(\alpha^2\xi, \alpha\eta),$$
$$\Phi_v(\xi, \eta) = \alpha\Phi'_v(\xi', \eta') = \alpha\Phi_v(\alpha^2\xi, \alpha\eta). \tag{5.14}$$

Furthermore, *after establishing* the relations (5.14) the value of the parameter $\alpha$ can be taken as equal to an arbitrary positive number, in particular

$$\alpha = \frac{1}{\sqrt{\xi}}.$$

Substituting this relation into (5.14), we obtain that *the determination of the functions $\Phi_u$, $\Phi_v$ of two variables is reduced to the determination of functions of a single variable*

$$\Phi_u(\xi, \eta) = \Phi_u\left(1, \frac{\eta}{\sqrt{\xi}}\right) = f_u\left(\frac{\eta}{\sqrt{\xi}}\right) = f_u\left(\frac{y}{\sqrt{\nu x/U}}\right)$$

and

$$\Phi_v(\xi, \eta) = \frac{1}{\sqrt{\xi}}\Phi_v\left(1, \frac{\eta}{\sqrt{\xi}}\right) = \frac{1}{\sqrt{\xi}} f_v\left(\frac{\eta}{\sqrt{\xi}}\right) = \sqrt{\frac{\nu}{Ux}}\, f_v\left(\frac{y}{\sqrt{\nu x/U}}\right). \tag{5.15}$$

As we see, the solution is self-similar. Thus the self-similarity of the solution to the boundary-layer problem (5.6), (5.7) is established and the expressions for the self-similar variables are obtained. However, it has been achieved as a result of the application of not only dimensional analysis but also the invariance of the problem with respect to an additional transformation group (5.13).

Introducing a new function

$$\varphi(\zeta) = \int_0^\zeta f_u(\zeta)\, d\zeta, \qquad \zeta = \frac{\eta}{\sqrt{\xi}} = \frac{y}{\sqrt{\nu x/U}},$$

we obtain from (5.6), (5.7) and the definition of the function $\varphi(\zeta)$ the relations

$$f_v = \frac{1}{2}(\zeta\varphi' - \varphi), \tag{5.16}$$

$$\varphi\varphi'' + 2\varphi''' = 0, \qquad \varphi(0) = \varphi'(0) = 0, \quad \varphi'(\infty) = 1; \tag{5.17}$$

here a prime indicates differentiation. The relationships (5.17) present a *boundary-value problem* for the ordinary equation $\varphi\varphi'' + 2\varphi''' = 0$, with boundary-value data at both $\zeta = 0$ and $\xi = \infty$. This is inconvenient, and here also a simple group-theoretical consideration is helpful. Indeed, let us consider the family of solutions to the equation $\varphi\varphi'' + 2\varphi''' = 0$ satisfying the *two boundary conditions* at $\zeta = 0$, $\varphi(0) = \varphi'(0) = 0$. It is easy to check that this family is invariant with respect to the transformation group:

$$\varphi_1(\zeta_1) = \alpha^{-1}\varphi(\zeta), \qquad \zeta_1 = \alpha\zeta, \tag{5.18}$$

so that if $\varphi(\zeta)$ is a solution to the equation $\varphi\varphi'' + 2\varphi''' = 0$ satisfying the first two boundary conditions in (5.17), then for any positive $\alpha$ the function $\alpha\varphi(\alpha\zeta)$ also satisfies the equation and these two boundary conditions.

Now consider the solution $\varphi_0(\zeta)$ to the Cauchy (not the boundary-value) problem for which the third boundary condition in (5.17), the condition at infinity, *is replaced by a condition at zero*, $\varphi_0''(0) = 1$. For the solution $\varphi_0(\zeta)$, which is easy to calculate numerically, the value of the derivative at infinity, $\varphi_0'(\infty)$, is 3.02. Therefore the solution $\varphi(\zeta) = \alpha\varphi_0(\alpha\zeta)$, where $\alpha = 1/\sqrt{2.086} = 0.6925$, satisfies all the conditions of problem (5.17).

For the drag $F$ on a section of unit width and length $l$ of the flat plate in a uniform stream of velocity $U$ we obtain from the previous relations, using the results of numerical calculation of the function $\varphi$,

$$\begin{aligned} F &= 2\int_0^l (\sigma_{xy})_{y=0}\,dx = 2U\sqrt{\frac{U}{\nu}}\,\rho\nu\int_0^l f_u'(0)\,\frac{dx}{\sqrt{x}} \\ &= 4\sqrt{\frac{U^3 l}{\nu}}\,\rho\nu\varphi''(0) = 4\alpha^3\rho\sqrt{U^3 l\nu} \\ &= 1.328\rho\sqrt{U^3 l\nu}. \end{aligned}$$

Here $(\sigma_{xy})_{y=0}$ is the shear stress on the plate.

Introducing the dimensionless parameter $\Pi = F/(\rho U^2 l)$ corresponding to the drag $F$, we get

$$\Pi = \Phi(\mathrm{Re}) = \frac{1.328}{\sqrt{Re}}, \qquad Re = \frac{Ul}{\nu}.$$

We note in passing that one can also see that at this well-known relation reveals incomplete similarity in the Reynolds number. In fact, the drag $F$ is determined by the following quantities: the length $l$ of the plate, the viscosity $\nu$ and density $\rho$ of the fluid and the velocity $U$ of the stream. Application of the standard procedure of dimensional analysis gives

$$\Pi = \Phi(Re).$$

For the high Reynolds numbers characteristic of the boundary layer there is no complete similarity with respect to Reynolds number, since there does not exist a non-zero limit of the function $\Phi = 1.328Re^{-1/2}$ as $Re \to \infty$. Hence the relations

$$\Pi = \text{const}, \qquad F = \text{const}\ \rho U^2 l$$

that would have to hold in the case of complete similarity in the Reynolds number cannot be expected to be true, no matter how high the Reynolds number. Nevertheless, one has the relation

$$\Pi^* = \frac{F}{\rho\sqrt{U^3 l\nu}} = \text{const} = 1.328,$$

corresponding to incomplete self-similarity: the parameter $\Pi^*$ cannot be obtained from standard dimensional analysis and contains the dimensional parameter $\nu$ whose explicit introduction into the problem violates self-similarity.

The example of boundary-layer flow past a flat plate which we have just considered is instructive also in the following aspect: the application of a more general group of transformations can here be given the form of *generalized dimensional analysis*, and this device turns out to be useful in many other special cases (but, it should be emphasized, not always).

Namely, we shall use different units to measure length in the $x$-direction and length in the $y$-direction. So, we introduce two different units of length, $l_x$ and $l_y$, and consider $x$ and $y$ as having different dimensions $L_x$ and $L_y$. Let us use this in the boundary-layer problem (5.6), (5.7). In this case all terms entering the boundary-layer equations and boundary conditions of the problem have identical dimensions if we take $[u] = [U] = L_x/T$, $[\nu] = L_y^2/T$, $[v] = L_y/T$, $[x] = L_x$ and $[y] = L_y$. Thus, among the governing parameters $v$, $x$, $U$ and $y$ not two but three have independent dimensions and the single independent dimensionless similarity parameter will be

$$\Pi_1' = \zeta = \frac{y}{\sqrt{\nu x/U}}, \tag{5.19}$$

whence follows immediately the self-similarity of the solution

$$u = Uf_u(\zeta), \qquad v = \sqrt{\nu U/x}\,f_v(\zeta). \tag{5.20}$$

It is instructive that using such independent units for longitudinal and transverse lengths is impossible for the full Navier–Stokes equations (5.5). In these equations the terms $\nu\partial^2_{yy}u$ and $\nu\partial^2_{yy}v$ appear in sum with the terms $\nu\partial^2_{xx}u$ and $\nu\partial^2_{xx}v$, so that if we measure $x$ and $y$ in different units these terms will have different dimensions, and this is impossible for equations having physical meaning. Consequently, the full Navier–Stokes equations, unlike the boundary layer equations, are not invariant with respect to the transformation group (5.13).

A natural question arises: is there an algorithm for seeking a maximally broad group of transformations with respect to which a given system of differential equations is invariant? Such an algorithm does exist. The basic ideas here belong to the Norwegian mathematician of the nineteenth century Sophus Lie. In recent times a series of general results and applications to particular systems of equations encountered in applied mathematics have been obtained; we refer the reader to the valuable books by Birkhof (1960), Bluman and Cole (1974) and Olver (1993).

## 5.3 The renormalization group and incomplete similarity

### 5.3.1 The renormalization group and intermediate asymptotics

Among the groups additional to the group of scaling transformations of quantities with independent dimensions that lead to scaling laws and self-similarity, a special and very important place belongs to the renormalization group. The renormalization group approach, following the ideas of Stückelberg and Peterman (1953), Gell-Mann and Low (1954), Bogolyubov and Shirkov (1955, 1959), Kadanoff (1966), Kadanoff *et al.* (1967), Patashinsky and Pokrovsky (1966) and Wilson (1971), has found extensive applications in modern theoretical physics. N. Goldenfeld, O. Martin and Y. Oono demonstrated a deep relation between the renormalization group method as traditionally used by physicists and the intermediate-asymptotics approach, developed independently and presented in this book. They did this by using the renormalization group method, in the form in which it is usually applied by physicists to solve some typical problems whose solution had been obtained previously by the method of intermediate asymptotics. Vice versa, they solved by the method of intermediate asymptotics some classical problems in statistical physics solved earlier by the renormalization group approach (Goldenfeld 1989; Goldenfeld, Martin and Oono 1989, 1991; Goldenfeld *et al.* 1990; Goldenfeld and Oono 1991; Chen, Goldenfeld and Oono 1991; Chen and Goldenfeld 1992; the book Goldenfeld 1992 is especially recommended).



We recall, see Chapters 1 and 4 and section 5.1 of this chapter, that any physically significant relation among dimensional (generally speaking) parameters

$$a = f(a_1, \dots a_k, b_1, \dots, b_m)$$

can be represented in the form of a relation between normalized dimensionless parameters $\Pi$, $\Pi_i$, $i = 1, \dots, m$:

$$\Pi = \Phi(\Pi_1, \dots, \Pi_m).$$

This is due to the compulsory invariance of physically significant relations with respect to the transformation group (5.3), (5.4) corresponding to a transition from the original system of units of measurement to an arbitrary system of units belonging to the same class of systems of units, i.e. having basic units of the same physical nature but different magnitude.

This means, we repeat, that every function $f$ which enters a physically significant relation possesses the property of generalized homogeneity:

$$f(a_1, \dots, a_k, b_1, \dots, b_m) = a_1^p \cdots a_k^r \Phi\left(\frac{b_1}{a_1^{p_1} \cdots a_k^{r_1}}, \dots, \frac{b_m}{a_1^{p_m} \cdots a_k^{r_m}}\right).$$

In the general case of incomplete similarity the function $\Phi$ possesses at large or small values of the dimensionless parameters $\Pi_{\ell+1}, \dots, \Pi_m$ the same property of generalized homogeneity in its own renormalized dimensionless arguments:

$$\begin{aligned}&\Phi(\Pi_1, \dots, \Pi_\ell, \Pi_{\ell+1}, \dots, \Pi_m)\\ &= \Pi_{\ell+1}^{\alpha_{\ell+1}} \cdots \Pi_m^{\alpha_m} \Phi_1\left(\frac{\Pi_1}{\Pi_{\ell+1}^{\beta_1} \cdots \Pi_m^{\delta_1}}, \dots, \frac{\Pi_\ell}{\Pi_{\ell+1}^{\beta_\ell} \cdots \Pi_m^{\delta_\ell}}\right)\end{aligned} \quad (5.21)$$

where the powers $\alpha_{\ell+1}, \dots, \delta_\ell$ are certain constants, which cannot be obtained by dimensional analysis even in principle.

The property of incomplete similarity also has a group-theoretical nature. It means that in addition to the compulsory group of transformations (5.3), (5.4) the problem at large or small values of the dimensionless parameters $\Pi_{\ell+1}, \dots, \Pi_m$ has the property of invariance with respect to the set of transformations

$$\begin{aligned} a_1' &= a_1, \quad a_2' = a_2, \quad \dots, \quad a_k' = a_k,\\ b_1' &= B_{\ell+1}^{\beta_1} \cdots B_m^{\delta_1} b_1, \quad \dots, \quad b_\ell' = B_{\ell+1}^{\beta_\ell} \cdots B_m^{\delta_\ell} b_\ell,\\ b_{\ell+1}' &= B_{\ell+1} b_{\ell+1}, \quad \dots, \quad b_m' = B_m b_m,\\ a' &= B_{\ell+1}^{\alpha_{\ell+1}} \cdots B_m^{\alpha_m} a. \end{aligned} \quad (5.22)$$

Here the parameters $B_{\ell+1}, \dots, B_m$ are certain positive numbers. Naturally, the values of these parameters should not be too large or too small, otherwise the applicability of the asymptotics (5.21) will be violated. The set (5.22) has the properties of a transformation group with parameters $B_{\ell+1}, \dots, B_m$. Indeed, if all the $B_{\ell+1}, \dots, B_m$ are equal to unity then the transformation (5.22) is an identity transformation. For every transformation in the set (5.22) there exists an inverse transformation with parameters $B_{\ell+1}^{-1}, \dots, B_m^{-1}$, also belonging to this set. Finally, the product of two transformations with parameters $B_{\ell+1}^{(1)}, \dots, B_m^{(1)}$ and $B_{\ell+1}^{(2)}, \dots, B_m^{(2)}$, which has parameters $B_{\ell+1} = B_{\ell+1}^{(1)} B_{\ell+1}^{(2)}, \dots, B_m = B_m^{(1)} B_m^{(2)}$, also exists in the set (5.22) and is uniquely determined. We will identify the group (5.22) with the renormalization group and so establish a link between this concept and the concepts of intermediate asymptotics and incomplete similarity considered earlier in this book.

More precisely, we will prove that the statement of the asymptotic invariance to the renormalization group (5.22) of the basic relation obtained after the application of dimensional analysis,

$$\Pi = \Phi(\Pi_1, \dots, \Pi_\ell, \Pi_{\ell+1}, \dots, \Pi_m), \tag{5.23}$$

is equivalent to the statement of incomplete similarity.

Indeed, assume that there is incomplete similarity in the parameters $\Pi_{\ell+1}, \dots, \Pi_m$ at small values, for definiteness sake, of these parameters, i.e. that the relation (5.21) holds for the function $\Phi$. Let us perform the transformations (5.22). We form the quantities

$$\Pi'_1 = \frac{b'_1}{a_1'^{p_1} \cdots a_k'^{r_1}} = B_{\ell+1}^{\beta_1} \cdots B_m^{\delta_1} \frac{b_1}{a_1^{p_1} \cdots a_k^{r_1}} = B_{\ell+1}^{\beta_1} \cdots B_m^{\delta_1} \Pi_1;$$

$$\vdots$$

$$\Pi'_\ell = \frac{b'_\ell}{a_1'^{p_\ell} \cdots a_k'^{r_\ell}} = B_{\ell+1}^{\beta_\ell} \cdots B_m^{\delta_\ell} \frac{b_\ell}{a_1^{p_\ell} \cdots a_k^{r_\ell}} = B_{\ell+1}^{\beta_\ell} \cdots B_m^{\delta_\ell} \Pi_\ell,$$

$$\Pi'_{\ell+1} = \frac{b'_{\ell+1}}{a_1'^{p_{\ell+1}} \cdots a_k'^{r_{\ell+1}}} = B_{\ell+1} \frac{b_{\ell+1}}{a_1^{p_{\ell+1}} \cdots a_k^{r_{\ell+1}}} = B_{\ell+1} \Pi_{\ell+1};$$

$$\vdots$$

$$\Pi'_m = \frac{b'_m}{a_1'^{p_m} \cdots a_k'^{r_m}} = B_m \frac{b_m}{a_1^{p_m} \cdots a_k^{r_m}} = B_m \Pi_m,$$

$$\Pi' = B_{\ell+1}^{\alpha_{\ell+1}} \cdots B_m^{\alpha_m} \frac{a}{a_1^{p} \cdots a_k^{r}} = \frac{a'}{a_1'^{p} \cdots a_k'^{r}} = B_{\ell+1}^{\alpha_{\ell+1}} \cdots B_m^{\alpha_m} \Pi.$$

Clearly, for every $i = 1, \dots, \ell$ we have by construction

$$\frac{\Pi'_i}{\Pi_{\ell+1}'^{\beta_i} \cdots \Pi_m'^{\delta_i}} = \frac{\Pi_i}{\Pi_{\ell+1}^{\beta_i} \cdots \Pi_m^{\delta_i}}$$

and also

$$\frac{\Pi'}{\Pi_{\ell+1}'^{\alpha_{\ell+1}} \cdots \Pi_m'^{\alpha_m}} = \frac{\Pi}{\Pi_{\ell+1}^{\alpha_{\ell+1}} \cdots \Pi_m^{\alpha_m}}.$$

We obtain using (5.21),

$$\begin{aligned}
\Pi' &= B_{\ell+1}^{\alpha_{\ell+1}} \cdots B_m^{\alpha_m} \Pi = B_{\ell+1}^{\alpha_{\ell+1}} \cdots B_m^{\alpha_m} \Phi(\Pi_1, \dots, \Pi_\ell, \Pi_{\ell+1}, \dots, \Pi_m) \\
&= B_{\ell+1}^{\alpha_{\ell+1}} \cdots B_m^{\alpha_m} \Pi_{\ell+1}^{\alpha_{\ell+1}} \cdots \Pi_m^{\alpha_m} \Phi_1 \left( \frac{\Pi_1}{\Pi_{\ell+1}^{\beta_1} \cdots \Pi_m^{\delta_1}}, \dots, \frac{\Pi_\ell}{\Pi_{\ell+1}^{\beta_\ell} \cdots \Pi_m^{\delta_\ell}} \right) \\
&= \Pi_{\ell+1}'^{\alpha_{\ell+1}} \cdots \Pi_m'^{\alpha_m} \Phi_1 \left( \frac{\Pi'_1}{\Pi_{\ell+1}'^{\beta_1} \cdots \Pi_m'^{\delta_1}}, \dots, \frac{\Pi'_\ell}{\Pi_{\ell+1}'^{\beta_\ell} \cdots \Pi_m'^{\delta_\ell}} \right) \\
&= \Phi(\Pi'_1, \dots \Pi'_\ell, \Pi'_{\ell+1}, \dots, \Pi'_m).
\end{aligned}$$

Thus, from incomplete similarity, (5.21), follows the invariance of the basic relation (5.23) with respect to the renormalization group (5.22). And now, vice versa, assume that there is invariance of the basic relation (5.23) with respect to the group (5.22). This means that for every $B_{\ell+1}, \dots, B_m$ the relation (5.23) preserves its form. Without loss of generality we can rewrite (5.23) in the form

$$\frac{\Pi'}{\Pi_{\ell+1}'^{\alpha_{\ell+1}} \cdots \Pi_m'^{\alpha_m}} = \Psi \left( \frac{\Pi'_1}{\Pi_{\ell+1}'^{\beta_1} \cdots \Pi_m'^{\delta_1}}, \dots, \frac{\Pi'_\ell}{\Pi_{\ell+1}'^{\beta_\ell} \cdots \Pi_m'^{\delta_\ell}}, \Pi'_{\ell+1}, \dots, \Pi'_m \right).$$

Returning to the previous variables we obtain

$$\begin{aligned}
\frac{\Pi'}{\Pi_{\ell+1}'^{\alpha_{\ell+1}} \cdots \Pi_m'^{\alpha_m}} &= \frac{\Pi}{\Pi_{\ell+1}^{\alpha_{\ell+1}} \cdots \Pi_m^{\alpha_m}} \\
&= \frac{\Phi(\Pi_1, \dots, \Pi_\ell, \Pi_{\ell+1}, \dots, \Pi_m)}{\Pi_{\ell+1}^{\alpha_{\ell+1}} \cdots \Pi_m^{\alpha}} \\
&= \Psi \left( \frac{\Pi_1}{\Pi_{\ell+1}^{\beta_1} \cdots \Pi_m^{\delta_1}}, \dots, \frac{\Pi_\ell}{\Pi_{\ell+1}^{\beta_\ell} \cdots \Pi_m^{\delta_\ell}}, B_{\ell+1}\Pi_{\ell+1}, \dots, B_m \Pi_m \right).
\end{aligned}$$

From this relation it follows (compare the proof of the basic theorem of dimensional analysis in Chapter 1) that the function $\Psi$ does not depend on the arguments $\Pi'_{\ell+1}, \dots, \Pi'_m$. Indeed, let us fix all parameters $B_i, i = \ell+1, \dots, m$, except for one, say $B_j$, and vary $B_j$ arbitrarily. The result will not depend on $B_j$.

Therefore

$$\Psi = \Phi_1\left(\frac{\Pi_1}{\Pi_{\ell+1}^{\beta_1}\cdots\Pi_m^{\delta_1}},\ldots,\frac{\Pi_\ell}{\Pi_{\ell+1}^{\beta_\ell}\cdots\Pi_m^{\delta_\ell}},\right).$$

Thus the function $\Phi$ has the property of generalized homogeneity (5.21) and we have a case of incomplete similarity. We have proved the equivalence of incomplete similarity and invariance with respect to the renormalization group.

### 5.3.2 The perturbation expansion

The basic relation (5.2) in which we are interested can be written in a dimensionless form as

$$\Pi = \Phi(\Pi_1,\ldots,\Pi_m,c).$$

Here we have added an additional constant dimensionless parameter $c$ on which the phenomenon is also assumed to depend. Its use will become clear shortly. Again let the parameters $\Pi_{\ell+1},\ldots\Pi_m$ be small for the sake of definiteness and assume that generally speaking, incomplete similarity holds, so that

$$\begin{aligned}\Pi &= \Phi(\Pi_1,\ldots,\Pi_\ell,\Pi_{\ell+1},\ldots,\Pi_m,c)\\ &= \Pi_{\ell+1}^{\alpha_{\ell+1}}\cdots\Pi_m^{\alpha_m}\Phi_1\left(\frac{\Pi_1}{\Pi_{\ell+1}^{\beta_1}\cdots\Pi_m^{\delta_1}},\ldots,\frac{\Pi_\ell}{\Pi_{\ell+1}^{\beta_\ell}\cdots\Pi_m^{\delta_\ell}},c\right);\end{aligned}$$

it follows that at least one of the powers $\alpha_{\ell+1},\ldots,\delta_\ell$ is different from zero. Generally speaking, $\alpha_{\ell+1},\ldots,\delta_\ell$ depend on the parameter $c$. Let us assume further that all the powers $\alpha_{\ell+1},\ldots,\delta_\ell$ vanish at $c=0$, i.e. that at $c=0$ we have a case of complete similarity . Then, for sufficiently small $\Pi_{\ell+1},\ldots,\Pi_m$, the function $\Phi(\Pi_1,\ldots,\Pi_\ell,\Pi_{\ell+1},\ldots,\Pi_m)$ can be replaced by its finite non-zero limit $\Phi(\Pi_1,\ldots,\Pi_\ell,0,\ldots,0)$, so that the dimensional parameters $b_{\ell+1},\ldots,b_m$ disappear from consideration. We can say, therefore, that at sufficiently small values for $\Pi_{\ell+1},\ldots,\Pi_m$ the phenomenon is asymptotically invariant to the transformation group

$$\begin{gathered}a'=a,\qquad a_1'=a_1,\qquad\ldots,\qquad a_k'=a_k,\\ b_1'=b_1,\quad\ldots,\quad b_\ell'=b_\ell;\quad b_{\ell+1}'=B_{\ell+1}b_{\ell+1},\quad\ldots,\quad b_m'=B_mb_m,\end{gathered}\tag{5.24}$$

where $B_{\ell+1},\ldots,B_m$ are the group parameters. In the case of incomplete similarity the problem is asymptotically invariant with respect to a more complicated renormalization group, (5.22).

The next step and, we emphasize, an independent one, is to obtain the parameters $\alpha_{\ell+1},\ldots,\delta_m$ by a perturbation expansion, using some quantitative relations concerning the phenomenon, in particular, the non-integrable conservation laws. The latter point is crucial: if no further information concerning the phenomenon under consideration is available then the parameters $\alpha_{\ell+1},\ldots,\delta_\ell$ entering the renormalization group (5.22) and the incomplete similarity relation (5.21) cannot be determined.

As an example we will consider a perturbation expansion for the problem of groundwater dome spreading with absorption considered in Chapter 3.

From the basic equation for the water head (3.5) the non-integrable (generally speaking) conservation law (3.8) was obtained:

$$\frac{d}{dt}\int_{-x_f}^{x_f}H(x,t)\,dx=-2\kappa c\int_{-x_f}^{x_f}(\partial_xH)^2dx.$$

The limiting self-similar solution was represented in the form (3.21):

$$\begin{aligned}H &= \frac{\xi_f^2\left(I\ell^{(1-3\mu)/\mu}\right)^{2\mu}\mu}{(\kappa t)^{1-2\mu}}\,f(\zeta,c),\\ \zeta &= \frac{x}{x_f},\qquad x_f=\xi_f\left(I\ell^{(1-3\mu)/\mu}\kappa t\right)^{\mu}.\end{aligned}\tag{5.25}$$

Therefore

$$\begin{aligned}\int_{-x_f}^{x_f}H(x,t)\,dx &= \frac{\xi_f^3\left(I\ell^{(1-3\mu)/\mu}\right)^{3\mu}\mu}{(\kappa t)^{1-3\mu}}\left(\int_{-1}^{1}f(\zeta,c)\,d\zeta\right),\\ \int_{-x_f}^{x_f}(\partial_xH)^2dx &= \frac{\xi_f^3\left(I\ell^{(1-3\mu)/\mu}\right)^{3\mu}\mu^2}{(\kappa t)^{2-3\mu}}\left(\int_{-1}^{1}[f'(\zeta,c)]^2d\zeta\right).\end{aligned}$$

At $c=0$, $\mu=1/3$; therefore the value of $1-3\mu$ is small at small values of $c$. To the accuracy of the leading terms we obtain

$$\begin{aligned}\frac{d}{dt}\int_{-x_f}^{x_f}H(x,t)\,dx &= \frac{1-3\mu}{3(\kappa t)^{2-3\mu}}\,\kappa\xi_f^3I\int_{-1}^{1}f(\zeta,0)\,d\zeta,\\ \int_{-x_f}^{x_f}(\partial_xH)^2dx &= \frac{\xi_f^3I}{9(\kappa t)^{2-3\mu}}\int_{-1}^{1}[f'(\zeta,0)]^2d\zeta.\end{aligned}$$

Now we need to use the relations

$$f(\zeta,0)=\frac{1}{4}(1-\zeta^2),\qquad f'(\zeta,0)=-\frac{1}{2}\zeta,$$

$$\int_{-1}^{1} f\,d\zeta=\frac{1}{3},\qquad \int_{-1}^{1}(f')^2 d\zeta=\frac{1}{6}.$$

We substitute these relations into the non-integrable conservation law (3.8) and obtain

$$-\tfrac{1}{9}(1-3\mu)=-\frac{1}{27}c,\qquad \text{so that}\qquad \mu=\tfrac{1}{3}\left(1-\frac{1}{3}c\right).$$

The same relation is obtained in the first approximation by expansion of the eigenvalue $\mu=(1-c)/(3-2c)$. This simple example illustrates the basic idea of the renormalization-group-with-perturbation-expansion approach. The following basic points should be noted. A scaling law, in our terms incomplete similarity, is assumed; this scaling law depends on a parameter. For the value zero of the parameter the solution is known. An asymptotic expansion is then used to find the solution for small but finite values of the parameter.

If there is no value of the parameter for which there exists complete similarity then the expansion cannot be performed. The only ways to obtain the 'anomalous dimensions', $\alpha_{\ell+1},\ldots,\delta_\ell$, are to solve the nonlinear eigenvalue problem or to perform an experiment, physical or numerical.

# Chapter 6

# Self-similar phenomena and travelling waves

## 6.1 Travelling waves

In various problems in applied mathematics an important role is played by *travelling waves*. These are phenomena for which distributions of the properties of motion at different times can be obtained from one another by a translation, rather than by a similarity transformation as in the case of self-similar phenomena. In other words, one can always choose a moving Cartesian coordinate system such that the distribution of properties of a phenomenon of travelling-wave type is stationary in that system.

In accordance with the definition given above, solutions of travelling-wave type can be expressed in the form

$$\mathbf{v}=\mathbf{V}(x-X(t))+\mathbf{V}_0(t). \tag{6.1}$$

Here $\mathbf{v}$ (generally speaking, a vector) is the property of the phenomenon being considered; $x$ is the spatial Cartesian coordinate, an independent variable of the problem; $t$ is another independent variable, for definiteness identified with time, although this is not necessary, and $X(t)$ and $\mathbf{V}_0(t)$ are time-dependent translations along the $x$- and $\mathbf{v}$-axes. In particular, if the properties of the process do not depend directly on time, so that the equations governing the process do not contain time explicitly, the travelling wave propagates uniformly:

$$\mathbf{v}=\mathbf{V}(x-\lambda t+c)+\mu t. \tag{6.2}$$

Here $\lambda$, $\mu$ and $c$ are constants; $c$ is the phase shift and $\lambda$ and $\mu$ represent the speeds of translation along the $x$- and $\mathbf{v}$-axes. For an important class of waves, steady travelling waves, the distribution of properties in a wave remains unchanged in time, so that $\mu=0$ and

$$\mathbf{v}=\mathbf{V}(x-\lambda t+c). \tag{6.3}$$